International Journal of Heat and Mass Transfer 51 (2008) 6312–6316

Contents lists available at ScienceDirect

International Journal of Heat and Mass Transfer

journal homepage: www.elsevier.com/locate/ijhmt

Technical Note

Comments on a paradox of viscous dissipation and its relation to the Oberbeck–Boussinesq approach

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ARTICLE INFO

Article history:
Received 31 October 2007
Available online 19 August 2008

Keywords:
Viscous dissipation
Buoyant flows
Porous media
Oberbeck–Boussinesq approximation
Natural convection

ABSTRACT

Some reflections are made about the object of a recent debate on the role played by the viscous dissipation term in the local energy balance equation. The debate regards both clear fluids and fluid saturated porous media. A strong accent has been placed on the role played by the pressure work term in the local energy balance. In the present brief contribution, it is suggested that the focus on the pressure work is not the most convincing escape from the possible paradoxes of viscous dissipation. On the other hand, another focus is proposed, namely the most appropriate formulation of the energy balance in the framework of the Oberbeck–Boussinesq approach to the description of buoyant flows.

1. Introduction

Two papers, [1,2], have opened a very interesting debate on the role played by the viscous dissipation term in the energy balance equation and its interplay with the pressure work term. In [1,2], the author describes a possible violation of the first law of thermodynamics in a problem of natural convection within a cavity either filled with a clear fluid or with a fluid saturated porous medium.

In a recently published paper [3], a thorough analysis of the effect of viscous dissipation in porous flows is performed. In this analysis, the cavity paradox described in Refs. [1,2] is revisited pointing out that some of the conclusions drawn in these papers are not completely correct. The common practice of neglecting the pressure work term is not the only reason for possible paradoxes of flows with viscous heating [3]. Another important actor appears in this scene: the Oberbeck–Boussinesq approximation. As it is clearly shown in [3], the role of viscous dissipation and pressure work must be carefully evaluated in a correct perspective of applicability of the Oberbeck–Boussinesq approximation.

The purpose of this short contribution is to add some further reflections on this matter, by showing that a different formulation of the local energy balance, the c_v-formulation, can be more suitable when the approximate Oberbeck–Boussinesq scheme is considered.

2. Resume of the cavity paradox of viscous dissipation

In Refs. [1,2], the author presents a paradox in two variants, depending on the reference to a clear fluid or to a fluid saturated porous medium. The paradox can be described, without specifying if we refer to a clear fluid or to a fluid saturated porous medium, as follows.

Consider a 2D cavity with impermeable boundaries (Fig. 1). The horizontal top and bottom boundaries are adiabatic, the left vertical wall is isothermal with temperature T_H, the right vertical wall is isothermal with temperature T_C, with $T_H > T_C$. There is a heat exchanged between the vertical walls corresponding in steady state to a power $\dot{Q}_{HC}$ flowing from the hot boundary at temperature T_H to the cold boundary at temperature T_C. A clockwise convection cell (or cells) will be established due to the density changes within the fluid. If we analyse this free convection flow according to the Oberbeck–Boussinesq approximation by taking into account the internal heating due to viscous dissipation, we would predict that the thermal power escaping the enclosure at the right vertical boundary is the sum of two terms: $\dot{Q}_{HC}$ and the power generated by viscous dissipation $\dot{Q}_{VD}$. On the other hand, the thermal power entering the enclosure at the left vertical boundary is merely $\dot{Q}_{HC}$. Therefore, since the enclosure represents a closed thermodynamic system we have a violation of the overall energy balance:

$$\dot{Q}_{HC} \neq \dot{Q}_{HC} + \dot{Q}_{VD}. \tag{1}$$

The author's reasoning goes on stating that the conceptual error that causes this violation of the energy balance is having deleted the pressure work term in the governing equations. In fact, this term would have produced a further power generated inside the fluid, $\dot{Q}_{PW}$. This power, unlike the power generated by viscous dissipation $\dot{Q}_{VD}$, would have been negative and such as to balance exactly $\dot{Q}_{VD}$. Therefore, the overall energy balance would have been fulfilled:

$$\dot{Q}_{HC} = \dot{Q}_{HC} + \dot{Q}_{VD} + \dot{Q}_{PW} \quad \text{with } \dot{Q}_{VD} + \dot{Q}_{PW} = 0.$$

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doi:10.1016/j.ijheatmasstransfer.2007.10.044

Nomenclature

$c_{\mathrm{p}}, c_{\mathrm{v}}$	specific heat at constant pressure, specific heat at constant volume
$\mathbf{g}$	gravitational acceleration
K	permeability
$\mathscr{L}$	operator depending on the momentum model
$\mathbf{n}$	unit normal vector
p	pressure
$\mathbf{q}$	heat flux density
$\dot{Q}_{\mathrm{C}}$	power escaping the cold boundary of the cavity
$\dot{Q}_{\mathrm{HC}}$	power entering the hot boundary of the cavity
$\dot{Q}_{\mathrm{PW}}$	power associated with pressure work
$\dot{Q}_{\mathrm{VD}}$	power generated by viscous dissipation
$\mathbf{s}$	strain tensor
T	temperature
T_0	reference temperature
$T_{\mathrm{H}}, T_{\mathrm{C}}$	boundary temperatures
$\mathbf{u}$	velocity
$\dot{W}$	mechanical power

Greek symbols

β	volumetric coefficient of thermal expansion
λ	thermodynamic coefficient of isochoric pressure change
μ, μ'	dynamic viscosity, effective dynamic viscosity
ρ	mass density
τ	viscous stress tensor

This is the essence of the point raised in Refs. [1,2].

The author's remark is correct, while the explanation based on the pressure work term is not convincing. As it is pointed out in [3], what is decidedly false is the sentence that he places (with negligible changes) in the Introduction of both Refs. [1,2]:

> …the main results and conclusions apply to any natural or mixed convection problem…

3. Why is the explanation based on the pressure work not convincing?

The cavity paradox is developed within the framework of the Oberbeck–Boussinesq approximation. As is well known, this approximation consists in neglecting density changes anywhere in the local balance equations except in the gravitational body force term in the momentum equation. As a consequence, not only we expect a violation of the energy balance, but also of the momentum and of the mass balances. In fact, all these three balances will be imperfect due to the approximation introduced. What is the extent of these violations? We don't know in general, since it depends on several variables the most important being certainly the externally prescribed temperature difference $T_{\mathrm{H}} - T_{\mathrm{C}}$. This difference is the driving cause of the convective flow and of the consequent viscous dissipation effect. The Oberbeck–Boussinesq approximation is better fulfilled the smaller is $T_{\mathrm{H}} - T_{\mathrm{C}}$. In particular, the smaller is $T_{\mathrm{H}} - T_{\mathrm{C}}$, the smaller is also the violation coefficient in the overall energy balance, namely the ratio $\dot{Q}_{\mathrm{VD}}/\dot{Q}_{\mathrm{HC}}$.

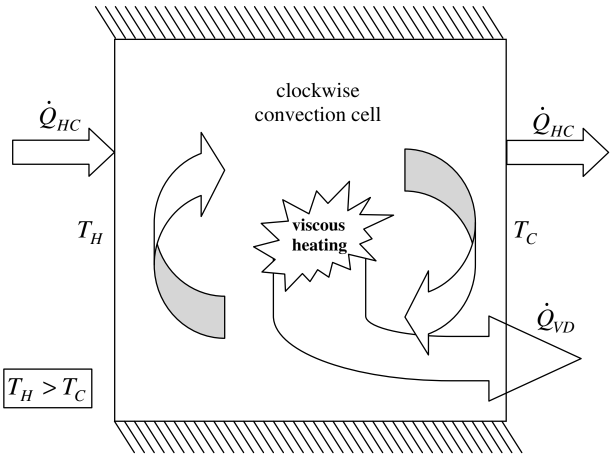

Fig. 1. Sketch of the enclosure.

In my opinion, the argument involving the pressure work term in the local energy balance is not convincing, for the following reason.

The exact local energy balance for a clear fluid can be expressed in two different formulations [4]

c_{v}-formulation :

$$\rho c_{\mathrm{v}} \frac{DT}{Dt} + p\lambda T \nabla \cdot \mathbf{u} = -\nabla \cdot \mathbf{q} + \tau : \mathbf{s}, \tag{2}$$

c_{p}-formulation :

$$\rho c_{\mathrm{p}} \frac{DT}{Dt} - \beta T \frac{Dp}{Dt} = -\nabla \cdot \mathbf{q} + \tau : \mathbf{s}, \tag{3}$$

where $D/Dt \equiv \partial/\partial t + \mathbf{u} \cdot \nabla$ is the convective derivative, p is pressure, $\mathbf{u}$ is velocity, $\mathbf{q}$ is the heat flux density vector, τ is the viscous stress tensor, $\mathbf{s}$ is the strain tensor, c_{p} and c_{v} are the specific heat at constant pressure and the specific heat at constant volume, respectively. The thermodynamic parameters λ and β are

$$\lambda = \frac{1}{p}\left(\frac{\partial p}{\partial T}\right)_{\rho}, \qquad \beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_{p}.$$

The term $\tau : \mathbf{s}$ is the power per unit volume generated by viscous dissipation. The term $\beta T Dp/Dt$ is the power per unit volume generated by pressure work. The above formulations of the local energy balance are general, since they don't contain approximations or hypotheses on the constitutive equations for the fluid. Note that the right hand sides in both formulations are identical.

Suppose that now I apply the Oberbeck–Boussinesq approximation in the c_{v}-formulation. The mass balance implies that $\mathbf{u}$ is solenoidal, $\nabla \cdot \mathbf{u} = 0$, so that

c_{v}-formulation [Oberbeck–Boussinesq approximation] :

$$\rho c_{\mathrm{v}} \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \tau : \mathbf{s}.$$

Viscous dissipation is everything one must deal with. Suppose that the flow is stationary, then one has

$$\rho c_{\mathrm{v}} \nabla \cdot (\mathbf{u}T) = -\nabla \cdot \mathbf{q} + \tau : \mathbf{s}. \tag{4}$$

The Oberbeck–Boussinesq approximation allows me to treat ρ as a constant and c_{v} as a constant as well, since the temperature differences felt by the fluid must be small. Thus, if I integrate Eq. (4) on the whole internal volume V of the enclosure, I obtain

$$0 = \dot{Q}_{\mathrm{HC}} - \dot{Q}_{\mathrm{C}} + \int_V \tau : \mathbf{s}\, \mathrm{d}V,$$

where

$$\dot{Q}_{\mathrm{HC}} = \int_{\text{hot boundary}} \mathbf{n} \cdot \mathbf{q} \, \mathrm{d}S,$$

$$\dot{Q}_{\mathrm{C}} = \int_{\text{cold boundary}} \mathbf{n} \cdot \mathbf{q} \, \mathrm{d}S,$$

and $\mathbf{n}$ is the horizontal unit vector oriented from the hot to the cold boundary. The vanishing of the volume integral of the left hand side of Eq. (4) is a consequence of the boundaries being impermeable. The volume integral of $\boldsymbol{\tau} : \mathbf{s}$ is precisely $\dot{Q}_{\mathrm{VD}}$. Therefore, I get

$$\dot{Q}_{\mathrm{C}} = \dot{Q}_{\mathrm{HC}} + \dot{Q}_{\mathrm{VD}}.$$

This equation provides the same information contained in Eq. (1). The heat escaping the cold boundary is higher than the heat entering the hot boundary. Obviously, the explanation of the paradox is not that proposed in Refs. [1,2], but the following one. I get a mismatch in the energy balance because I have neglected, following the Oberbeck–Boussinesq approximation, a term proportional to $\nabla \cdot \mathbf{u}$ (that is not exactly zero) and I have considered both ρ and c_{v} as constants (that is not exactly true). The conclusion can be stated in the form of a question: Why should I expect an approximate theory to fulfil exactly an overall (mass, momentum, energy) balance?

4. Why does not the cavity paradox apply to every natural or mixed convection problem?

As stated at the end of Section 2, in Refs. [1,2] it is claimed that the cavity paradox can be extended to any natural or mixed convection problem. This is not true. It can be formulated if and only if one deals with an enclosure having impermeable boundaries. In other words, it applies if and only if the fluid system is a closed thermodynamic system. If one has a flow system as, for instance, a channel or duct (vertical, inclined or horizontal), the reasoning is completely different.

I refer to the scheme in Fig. 2. I imagine a horizontal channel with impermeable lateral boundaries, where the fluid enters with a given mass flow rate (in steady regime) and exits with the same mass flow rate. If the lateral boundaries are thermally insulated and I take into account viscous dissipation, the fluid will enter with a temperature T_{C} smaller than the exit temperature T_{H}, the difference being due to the viscous heating. As in the case of the enclosure, the heat crossing the inlet section, i.e. the power $\dot{Q}_{\mathrm{HC}}$, does not coincide with the power crossing the outlet section, i.e. the power $\dot{Q}_{\mathrm{HC}} + \dot{Q}_{\mathrm{VD}}$. However, here the mismatch does not imply any paradox as in the case of the enclosure, since the overall energy balance is fulfilled. In fact, to ensure a steady mass flow through the channel, one must employ a mechanical device (a pump) that provides a mechanical power contribution $\dot{W}$ into the flow system. Thus, one has

$$\dot{Q}_{\mathrm{HC}} + \dot{W} = \dot{Q}_{\mathrm{HC}} + \dot{Q}_{\mathrm{VD}}.$$

This energy balance is closed if $\dot{W} = \dot{Q}_{\mathrm{VD}}$, i.e. if the power generated by viscous dissipation is perfectly balanced by the mechanical power produced by the pump. In this case, the reasoning performed in Refs. [1,2] does not lead to any apparent paradox. Obviously, if buoyancy is taken into account and if the Oberbeck–Boussinesq approximation is used, one should not expect the equality $\dot{W} = \dot{Q}_{\mathrm{VD}}$ to be perfectly satisfied.

5. The overall pressure work in a cavity

Let us assume that the fluid is contained in an enclosure bounded by impermeable walls, whose internal volume is V. I consider Eqs. (2) and (3) referred to a steady process. If I subtract Eq. (2) from Eq. (3), I obtain

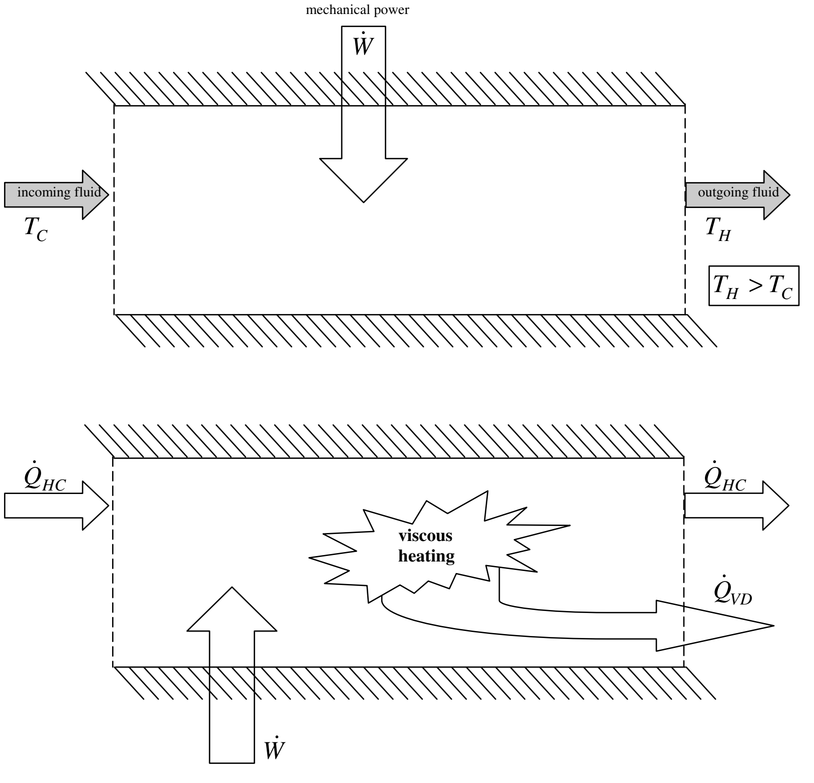

Fig. 2. Sketch of an open flow system.

$$\rho(c_p - c_v)\mathbf{u}\cdot\nabla T - \beta T\mathbf{u}\cdot\nabla p - p\lambda T\nabla\cdot\mathbf{u} = 0. \tag{5}$$

The term $\beta T\mathbf{u}\cdot\nabla p$ is the pressure work term. I invoke the Oberbeck–Boussinesq approximation, so that ρ is constant both in the mass balance and in the energy balance, while $\nabla\cdot\mathbf{u} = 0$. Then Eq. (5) can be rewritten as

$$\rho(c_p - c_v)\nabla\cdot(\mathbf{u}T) = \beta T\mathbf{u}\cdot\nabla p. \tag{6}$$

As stated in the previous sections, the Oberbeck–Boussinesq approximation implies that the temperature range within the domain V occupied by the fluid must be small, so that it appears as reasonable to assume that both c_p and c_v are constant. Thus, if I integrate Eq. (6) in the domain V, I get

$$\rho(c_p - c_v)\int_V \nabla\cdot(\mathbf{u}T)\,\mathrm{d}V = \int_V \beta T\mathbf{u}\cdot\nabla p\,\mathrm{d}V. \tag{7}$$

Since the boundary of V is impermeable, the integral on the left hand side of Eq. (7) is zero. Therefore, the integral of the pressure work term in a cavity with impermeable walls must be zero. This conclusion rules out the conceivability of the reasoning appearing in Refs. [1,2] on the balance between the overall viscous dissipation contribution and the pressure work contribution, described in Section 2.

6. The Oberbeck–Boussinesq approximation, the pressure work and the reference temperature

Let us consider the stationary local momentum balance equation, according to the Oberbeck–Boussinesq approximation,

$$\mathscr{L}\{\mathbf{u}\} - \rho\mathbf{g}[1 - \beta(T - T_0)] = -\nabla p, \tag{8}$$

where p is the pressure and $\mathscr{L}$ is an operator depending on the special model considered, for instance,

$$\begin{aligned}
&\text{Clear fluid}: \quad \mathscr{L}\{\mathbf{u}\} \equiv \rho(\mathbf{u}\cdot\nabla)\mathbf{u} - \mu\nabla^2\mathbf{u},\\
&\text{Darcy porous flow}: \quad \mathscr{L}\{\mathbf{u}\} \equiv \frac{\mu}{K}\mathbf{u},\\
&\text{Brinkman porous flow}: \quad \mathscr{L}\{\mathbf{u}\} \equiv \frac{\mu}{K}\mathbf{u} - \mu'\nabla^2\mathbf{u}.
\end{aligned}$$

All the properties ρ, μ, μ', β, K are constants. The constant temperature T_0 is the reference temperature required by the Oberbeck–Boussinesq approximation. The explicit dependence on T_0 is an unpleasant feature, since seemingly the solution depends explicitly on T_0. However, this explicit dependence is only an apparent one. In fact, if one takes the curl of Eq. (8), one obtains

$$\nabla\times(\mathscr{L}\{\mathbf{u}\} + \rho\mathbf{g}\beta T) = 0, \tag{9}$$

where the explicit dependence on T_0 has disappeared and the pressure field p is no more part of the solution.

If one considers the local energy balance Eq. (3) with reference to a stationary flow, one notices that the pressure work term

$$\text{pressure work} = \beta T\mathbf{u}\cdot\nabla p$$

is the only term where the reference temperature arises again explicitly. In fact, on account of Eq. (8), one has

$$\begin{aligned}
\text{pressure work} &= -\beta T\mathbf{u}\cdot\{\mathscr{L}\{\mathbf{u}\} - \rho\mathbf{g}[1 - \beta(T - T_0)]\}\\
&= -\beta T\mathbf{u}\cdot\mathscr{L}\{\mathbf{u}\} + \rho\beta T\mathbf{u}\cdot\mathbf{g} - \rho\beta^2T^2\mathbf{u}\cdot\mathbf{g}\\
&\quad + \rho\beta^2TT_0\mathbf{u}\cdot\mathbf{g}.
\end{aligned} \tag{10}$$

The value of the term $\rho\beta^2TT_0\mathbf{u}\cdot\mathbf{g}$ is completely unknown, unless T_0 is precisely determined.

It must be pointed out that, with or without the pressure work term, the Oberbeck–Boussinesq local balance equations contain an implicit dependence on the reference temperature T_0. Indeed, the values one should assign to all the constant properties ρ, μ, μ', β, K, c_p refer to the temperature T_0. However, this implicit dependence is a negligible one since, as long as the Oberbeck–Boussinesq approximation applies, all these properties have a poor dependence on T_0. Thus, one can forget this dependence on T_0 inasmuch as one neglects it in solving an isothermal flow problem or a heat conduction problem.

According to the definitions given above, in order to yield a sound physical theory, the Oberbeck–Boussinesq local balance equations should not contain any explicit dependence on T_0. Therefore, one could adopt the c_p-formulation of the local energy balance and neglect the pressure work term, or adopt the c_v-formulation. In practice, both choices would lead formally to the same equation except for the kind of specific heat (c_p or c_v). In the case of liquid flows, this appears as a minor problem since most thermodynamic tables report the data of "specific heat" without distinguishing between c_p and c_v. For liquid water at saturation pressure, one has [5]

$$\begin{aligned}
&20\,^\circ\mathrm{C}: \quad \frac{c_p - c_v}{c_p} \cong 0.6\%,\\
&50\,^\circ\mathrm{C}: \quad \frac{c_p - c_v}{c_p} \cong 3\%.
\end{aligned}$$

These discrepancies appear as reasonably acceptable in an approximate theory (Oberbeck–Boussinesq) where several sharp assumptions have been made.

A special case where the pressure work term does not depend explicitly on T_0 and where this contribution can be easily compared with the viscous dissipation contribution is Darcy parallel horizontal flow. In this case, one has $\mathbf{u}\cdot\mathbf{g} = 0$, so that the pressure work term, Eq. (10), is given by

$$\text{pressure work} = -\beta T\mathbf{u}\cdot\mathscr{L}\{\mathbf{u}\} = -\beta T\frac{\mu}{K}\mathbf{u}\cdot\mathbf{u},$$

while the viscous dissipation term is

$$\text{viscous dissipation} = \frac{\mu}{K}\mathbf{u}\cdot\mathbf{u}.$$

One has

$$\left|\frac{\text{pressure work}}{\text{viscous dissipation}}\right| = \beta T.$$

For liquid water at saturation pressure, one has

$$\begin{aligned}
&20\,^\circ\mathrm{C}: \quad \beta T_0 \cong 0.06,\\
&50\,^\circ\mathrm{C}: \quad \beta T_0 \cong 0.14.
\end{aligned}$$

For an ideal gas, one has $\beta = 1/T_0$. In this case, one would have a perfect balance between viscous dissipation and pressure work at those positions where $T = T_0$.

7. Conclusions

- The paradox of the "energy generating" cavity pointed out in Refs. [1,2] is a consequence of the Oberbeck–Boussinesq approximation. The explanation based on the non-negligible contribution of pressure work in the energy balance is not convincing. In fact, according to the Oberbeck–Boussinesq approximation, the integral of the pressure work term in a cavity with impermeable walls must be zero.
- The c_v-formulation of the local energy balance appears to be the simplest one, when the Oberbeck–Boussinesq approximation is adopted.

- The difference between the c_v-formulation and the c_p-formulation of the local energy balance with no pressure work term relies only in the different values of c_v and c_p. The relative discrepancy between these values for liquids is small.
- The c_p-formulation of the local energy balance with the pressure work term has intrinsically a problem of explicit dependence on the choice of the reference temperature T_0, when the Oberbeck–Boussinesq approximation is adopted.

Acknowledgements

I am deeply grateful to Dr. Andrew Rees, University of Bath (UK), for having encouraged me to start this work. I am also grateful to Prof. Enzo Zanchini, a University of Bologna colleague, for the helpful discussions on the thermodynamic aspects of this subject.

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